Solving a set of linear equations of the form *Ax = b* using iterative methods

Jacobi Method ('simultaneous displacements')

A = Matrix, x = vector (solving for x = {x*1*, x*2*, x*3*….x*n*), b = vector
Iteration 1: Start with an initial guess for x*1*,x*2*…x*n*
Using values of x*i* from the initial guess find new values of x1,x2, ….xn, by solving equation i for xi
Iteration 2: Use the new values of X obtained from the previous step as the next guess and repeat
Assumptions: There exists a unique solution and the sufficient condition convergence is satisfied. (loosely, all diagonal elements are non-zero)

Convergence criteria:
A sufficient (but not necessary) condition for the method to converge is that the matrix *A* is strictly or irreducibly diagonally dominant. Strict row diagonal dominance means that for each row, the absolute value of the diagonal term is greater than the sum of absolute values of other terms:

When should I stop? How many iterations?

Define NUM_ITER = 10000 (or a sufficiently large number)

In addition, check for the L2 norm ||X*n-1* – X*n*|| <= 0.01 (or any small number)

Continue processing the next iteration if the difference between 2 successive iterations is not significantly small.

7.3 The Jacobi and Gauss-Seidel Iterative Methods

The Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

Has a unique solution.

2. The coefficient matrix *A* has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation,* $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approximation's x-vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots x_n^{(k)}\right)^t, \quad k = 1,2,3, \dots$

Gauss-Seidel method ('successive displacements') The Gauss-Seidel method is a variant of the Jacobi method that usually improves the rate of convergence.